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Charged singularities: repulsive effects

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Abstract. The repulsive phenomena which a particle experiences in the vicinity of a naked singularity are investigated in the Kerr–Newman space–time. The aim is to extend the knowledge of this fact to charged solutions and to have a direct indication of how, in these situations, the gravitational and electrostatic interactions are competing.

1. Introduction

Repulsive gravitational effects are known to occur in the context of general relativity. A black hole, for example, tends to preserve its state repelling any test field which would destroy its event horizon: in this case repulsive forces are produced of either an electrostatic or a spin–spin kind (Wald 1972, de Felice 1974).

More remarkable are the repulsive phenomena which take place near a naked singularity; they have been extensively investigated in the case of the Reissner–Nordström and the Kerr naked singularity solutions. Here, the physical interpretation of the repulsion is not so immediate as in the black-hole case; different interpretations have in fact been produced (Israel 1970, de Felice 1974, Cohen and Gautreau 1979).

For the sake of completeness we would like here to extend this investigation to the Kerr–Newman (hereafter K–N) naked singularity solution; in this case the combined effects of charge and angular momentum are expected to give rise to more interesting situations.

2. The equations of motion

The Kerr–Newman solution of Einstein's equations is the charged generalisation of the Kerr solution and reads, in Boyer–Lindquist coordinates (Boyer and Lindquist 1967),

$$ds^2 = -(\Delta/\Sigma)(dt - a\sin^2\theta\, d\phi)^2 + (\Sigma^{-1}\sin^2\theta)[(r^2+a^2)\, d\phi - a\, dt]^2 + (\Sigma/\Delta)\, dr^2 + \Sigma\, d\theta^2 \tag{1}$$

where

$$\Delta = r^2 + a^2 + Q^2 - 2Mr \qquad \Sigma = r^2 + a^2\cos^2\theta. \tag{2}$$

Here M is the mass of the source, a its specific angular momentum and Q its charge.

As is well known, this metric describes a charged and rotating black hole when $(a^2+Q^2) \leqslant M^2$, while it describes a naked singularity when $(a^2+Q^2) > M^2$; in the following we shall assume that the latter inequality is satisfied, and therefore $\Delta > 0$

always. As one can see from (2), the charge Q enters the function Δ only, so that the curvature singularity, which occurs where $\Sigma = 0$, maintains the ring-like structure as in the Kerr metric.

Because of the electromagnetic interaction, a test particle of mass μ and charge e will not move along a geodesic; nevertheless, the equations of motion have been completely solved and read (Carter 1968)

$$\Sigma\, dt/d\lambda = -a(aE \sin^2\theta - l) + (r^2+a^2)P/\Delta \tag{3}$$

$$\Sigma\, dr/d\lambda = \pm\{P^2 - \Delta[\mu^2 r^2 + (l-aE)^2 + (L-l^2)]\}^{1/2} \tag{4}$$

$$\Sigma\, d\theta/d\lambda = \pm\{(L-l^2) - \cos^2\theta[a^2(\mu^2-E^2) + l^2/\sin^2\theta]\}^{1/2} \tag{5}$$

$$\Sigma\, d\phi/d\lambda = -(aE - l/\sin^2\theta) + aP/\Delta \tag{6}$$

where $P = E(r^2+a^2) - al - eQr$. Here E is the energy of the particle at infinity, l is the axial angular momentum and L is related in a non-trivial way (de Felice 1980) to the square of the total angular momentum. These quantities are constants of motion: E and l arise from the explicit symmetries of the metric (stationarity and axisymmetry) while L arises from the separability of the Hamilton–Jacobi equation.

Let us note that the angular equation of motion for θ, equation (5), is independent of the charge Q; therefore the description of the θ-motion in the K–N metric remains the same as in the Kerr one. In particular, as in that case, with the aid of equation (6), one can make a distinction between an 'orbital' and a 'vortical' type of motion (de Felice and Calvani 1972, Bičák and Stuchlík 1976); in the former, the condition $(L-l^2)>0$ is satisfied and the test particle can cross the equatorial plane; in the latter, the motion is confined between two $\theta=$ constant hyperboloids. This occurs when the constants of motion satisfy the inequalities

$$|L| < a^2\Gamma \qquad (\Gamma = E^2 - \mu^2) \tag{7a}$$

$$L < l^2 \leq (L + 4a^2\Gamma)^2/4a^2\Gamma. \tag{7b}$$

In (7b) the equality sign corresponds to the generalised radial trajectories ($\theta =$ constant $\neq \pi/2$). Solving (7b) with respect to L, the vortical conditions become

$$L \geq -a^2\Gamma + 2al\sqrt{\Gamma} \qquad (l>0)$$

$$L \geq -a^2\Gamma - 2al\sqrt{\Gamma} \qquad (l<0).$$

In the following we shall be interested in those vortical orbits which are confined on the hyperboloids $\theta =$ constant (generalised radial orbits), because in this case the repulsive phenomena are more transparent; the conditions for such a motion are, from equations (5) and (7),

$$\begin{aligned} l &= \epsilon a\sqrt{\Gamma}\sin^2\theta \\ L &= -\Gamma a^2 \cos 2\theta. \end{aligned} \qquad \epsilon = \pm 1, \Gamma > 0. \tag{8}$$

3. The effective radial potential

The reality condition upon equation (4) leads to the following relation for the particle energy:

$$\alpha E^2 - 2\beta E + \gamma \geq 0 \tag{9}$$

where

$$\alpha = (r^2+a^2)^2 - \Delta a^2$$
$$\beta = (2Mr - Q^2)al + eQr(r^2+a^2)$$
$$\gamma = a^2l^2 + 2eQral + Q^2r^2e^2 - \Delta(\mu^2r^2+L). \tag{10}$$

Qualitiative features of the radial motion can be deduced from the effective radial potential $E_+(r)$ which is the positive root of the equation associated with (9). It is easy to prove that a test particle moving outside the equatorial plane never reaches the ring singularity ($r=0$, $\theta=\pi/2$). The value of the effective potential at $r=0$ is given by

$$E_0 = -\frac{Q^2al + [(l^2-L)(a^2+Q^2)]^{1/2}}{a^2Q^2}. \tag{11}$$

We first notice that it does not depend on the charge e of the particle; therefore the considerations which follow apply to all particles, regardless of their charge.

Let us consider the orbital motion; the condition $L > l^2$ holds, so that E_0 is not real†. From equations (9) and (10) it follows that at $r \geq 0$, $(\mathrm{d}r/\mathrm{d}\lambda)^2 < 0$ and therefore the particle must meet a turning point (where $\mathrm{d}r/\mathrm{d}\lambda = 0$) before reaching the singularity.

In the vortical motion case $L < l^2$ so that E_0 is real; this means that a particle can reach $r=0$, but from equation (5) one knows that this happens only with $\theta \neq \pi/2$. Those particles therefore never reach the singularity either.

As in the Kerr metric, here also the ring singularity could be reached by particles moving strictly in the equatorial plane; a detailed study of the equatorial motion in the K–N metric has not been carried out so far (Sharp 1979).

4. Repulsive barriers

In previous works (de Felice *et al* 1975, Calvani and de Felice 1978, Calvani *et al* 1978) it was pointed out that the most significant way to study the occurrence and the nature of the repulsive phenomena which take place near a naked singularity is to consider the vortical orbits, equations (7), and in particular the generalised radial orbits, equation (8). These in fact would never cross the equatorial plane and, unless they met a turning point, they would cross the ring $r=0$ and get lost in the $r<0$ sheet of the metric which is of dubious physical meaning. For these particles a turning point is indeed a point of repulsion because their orbit remains entirely confined above or below the equatorial plane (figure 1).

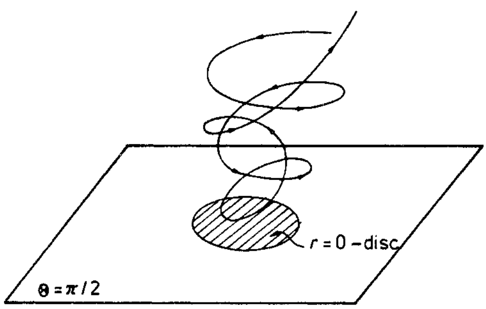

Figure 1. Qualitative picture of the vortical motion; the ring denotes the ring singularity in the equatorial plane.

† When $L = l^2$ the motion is in the equatorial plane $\theta = \pi/2$, but it has to be treated separately.

In the Kerr metric the most evident case of repulsion was shown on those particles which move on stable trajectories on the hyperboloid $\theta = \text{constant}$, with parameters

$$E/\mu = 1 \qquad l = L = 0$$

(parabolic orbits). They approach the source from infinity down to the $r = 0$ disc, which is the surface bounded by the ring singularity. Here they stop and go back to the same asymptotic region they came from (figure 1).

The $r = 0$ disc ($r = 0$, $\theta \neq \pi/2$) is a rather peculiar surface; there in fact the Kerr metric acquires the flat space–time form, without being flat (the curvature does not vanish). A particle sitting there would be pushed off to (positive) infinity with a definite acceleration.

A similar situation arises in the K–N space–time but on the surface

$$r = Q^2/2M \equiv r_*. \tag{12}$$

There again the metric acquires the flat space–time form without being flat (Misner *et al* 1973 p 903). This surface moreover is only found in the $r > 0$ sheet and contains the ring singularity in its interior.

Let us now consider the repulsive barriers for the generalised radial orbits only; these are found by looking for the zeros of the radial equation (4) with the conditions (8). We have then

$$\begin{aligned} a^4 \sin^4 \theta - 2a^2 \sin^2 \theta[(r^2 + a^2) + \epsilon(G - \epsilon)(2Mr - Q^2) - \epsilon\beta r(G^2 - 1)^{1/2}] \\ + (r^2 + a^2)[(r^2 + a^2) + (G^2 - 1)(2Mr - Q^2) \\ - 2\beta r G(G^2 - 1)^{1/2}] + \beta^2 r^2 (G^2 - 1) = 0 \end{aligned} \tag{13}$$

where we have put

$$\mu = 1, \qquad \Gamma = E^2 - 1, \qquad G = \left(\frac{\Gamma + 1}{\Gamma}\right)^{1/2}, \qquad \beta = eQ.$$

Solving with respect to $\Phi^2 = \sin^2 \theta$ we have

$$\Phi_\pm^2 = a^{-2}[(r^2 + a^2) + \epsilon(G - \epsilon)(2Mr - Q^2) - \epsilon\beta r(G^2 - 1)^{1/2} \pm \delta^{1/2}] \tag{14}$$

where

$$\delta = \Delta[(G - \epsilon)^2(Q^2 - 2Mr) + 2\beta r(G - \epsilon)(G^2 - 1)^{1/2}]. \tag{15}$$

Due to the complexity of the solution (14), we shall deal with the various cases separately.

(a) $\beta = 0$.

This is the simplest situation, but also the most instructive, because it describes the pure gravitational repulsive effects. The reality condition on $\Phi_\pm^2$ requires that

$$\delta = \Delta(G - \epsilon)^2(Q^2 - 2Mr) \geqslant 0. \tag{16}$$

This shows that repulsive barriers, if any, occur only at $r \leqslant r_*$ (we recall that $\Delta > 0$ for a naked singularity).

Solution (14) now reads

$$\Phi_\pm^2 = a^{-2}\{(r^2 + a^2) + \epsilon(G - \epsilon)(2Mr - Q^2) \pm (G - \epsilon)[\Delta(Q^2 - 2Mr)]^{1/2}\} \qquad (\beta = 0) \tag{17}$$

and must satisfy the obvious conditions

$$0 \leq \Phi_{\pm}^2 \leq 1. \tag{18}$$

One immediately sees that on the surface $r = r_*$ it is

$$\Phi_+^2 = \Phi_-^2 = (r_*^2 + a^2)/a^2 > 1; \tag{19}$$

moreover, the solution Φ_+^2 is always greater than one, whatever ϵ is. Therefore the only solution of interest is Φ_-^2, which can be written as

$$\Phi_-^2 = (Q^2 - 2Mr)a^{-2}[\eta - \epsilon G - (G - \epsilon)\eta^{1/2}] \qquad (\beta = 0) \tag{20}$$

where

$$\eta = \Delta/(Q^2 - 2Mr). \tag{21}$$

To look for the zeros of (20) let us further define $\xi^2 = \eta$, so that the zeros of (20) are solutions of

$$\xi^2 - (G - \epsilon)\xi - \epsilon G = 0, \tag{22}$$

that is

$$\xi_\pm = \tfrac{1}{2}[(G - \epsilon) \pm (G + \epsilon)]. \tag{23}$$

One then has

$$\xi_+ = G > 1, \qquad \xi_- = -\epsilon, \tag{24}$$

but, as $\xi > 1$, the only possible solution is $\xi_+ = G$. The locus of zeros for Φ_-^2 is therefore along the curve

$$G^2 = \Delta/(Q^2 - 2Mr) \tag{25}$$

and is shown in figure 2(a). Note that G^2 does not depend on ϵ and is always greater than one; the minimum is at $r = [Q^2 - (Q^4 + 4a^2M^2)^{1/2}]/2M$.

With the help of figure 2(a) and condition (19) it is now straightforward to draw, for a given value of G, the repulsive barriers, equation (20), as shown in figures 2(b) and 2(c). Note that from figure 2(a) one deduces that when $G^2 > (1 + a^2/Q^2)$, the curve Φ_-^2 must cross the r-axis at a positive value of r, so that $\Phi_-^2(r = 0)$ must be negative; while for $1 < G^2 < (1 + a^2/Q^2)$, $\Phi_-^2(r = 0)$ is positive; moreover, for $1 < G^2 < G_{\mathrm{MIN}}$ the curve Φ_-^2 has no zeros.

In figures 2(b) and 2(c) the full curve is for $\epsilon = +1$ ($l > 0$) and the broken one for $\epsilon = -1$ ($l < 0$); it is in fact easy to prove that

$$\Phi_-^2(l > 0) \geq \Phi_-^2(l < 0),$$

the equality sign holding on $r = r_*$ and the roots of (25). As one expects, particles with high enough energy succeed in overcoming the repulsive barriers (see figures 2(a) and 2(c)).

(b) $\beta \neq 0$.

In this case the discriminant of equation (14) can be written as

$$\delta = \Delta(G - \epsilon)^2\left[(Q^2 - 2Mr) + 2\beta r\left(\frac{G + \epsilon}{G - \epsilon}\right)^{1/2}\right] \tag{26}$$

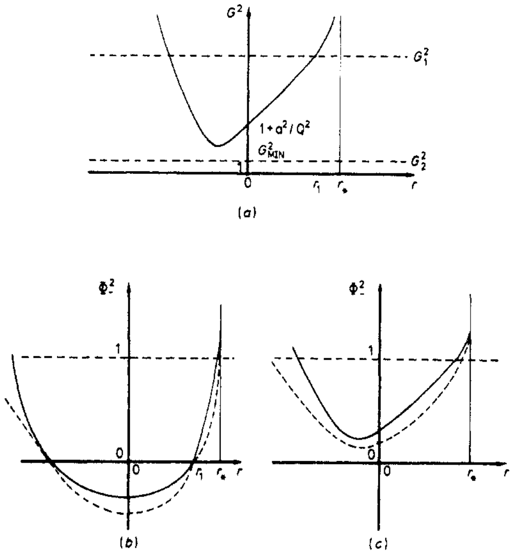

Figure 2. (*a*) The function (25) is shown; it gives the zeros of Φ^2_- ($\beta = 0$) for a chosen value of G. r_1 is a zero of Φ^2_- for $G = G_1$, while Φ^2_- has no zeros for $G = G_2$. (*b*) The radial barriers are shown for $G = G_1$; the full curve is for $l > 0$ and the broken one for $l < 0$. A particle coming in from infinity and moving on a hyperboloid θ = constant finds an inversion point for any θ. (*c*) The radial barriers are shown for $G = G_2$; for certain values of θ, i.e. of l and L through (8), there are no inversion points, and the particle goes in the $r < 0$ sheet.

and the reality condition is

$$h \equiv \left(\frac{G+\epsilon}{G-\epsilon}\right)^{1/2} \geqslant \frac{2Mr - Q^2}{2\beta r}. \tag{27}$$

This function is drawn in figure 3(*a*) for $\beta > 0$ and in figure 3(*b*) for $\beta < 0$. In the shaded part $\delta < 0$. As the following limits hold:

$$\epsilon = +1 \qquad (l > 0): \qquad \lim_{\Gamma \to \infty} h = \infty \qquad \lim_{\Gamma \to 0} h = 1$$

$$\epsilon = -1 \qquad (l < 0): \qquad \lim_{\Gamma \to \infty} h = 0 \qquad \lim_{\Gamma \to 0} h = 1$$

it is possible to see in a qualitative way, with the help of figure 3, how the angular momentum and the charge of the test particle act to modify the repulsive barriers with respect to the uncharged case.

In order to have more information about $\Phi^2_\pm$, let us define

$$\tilde{\eta} = Q^2 - 2Mr + 2\beta rh; \tag{28}$$

the solution then becomes

$$\Phi^2_\pm = a^{-2}\{(r^2 + a^2) + (G - \epsilon)[\epsilon(\beta rh - \tilde{\eta}) \pm (\Delta\tilde{\eta})^{1/2}]\}. \tag{29}$$

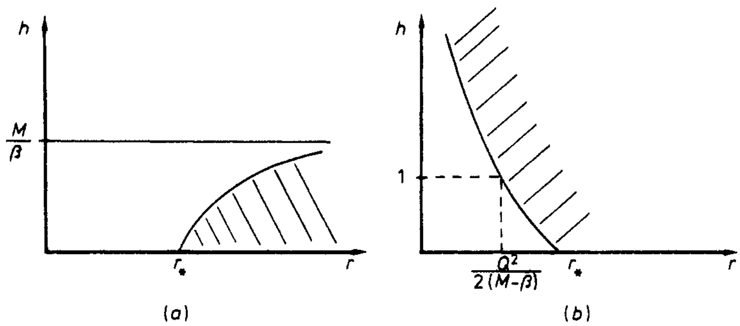

Figure 3. The function (27) is shown. For a chosen value of G it gives the zeros of the discriminant (26), i.e. where $\Phi_-^2 = \Phi_+^2$ for $\beta \neq 0$. Case (a) is for $\beta > 0$ and case (b) for $\beta < 0$.

Let $\tilde{r}$ be the solution of $\tilde{\eta} = 0$, i.e.

$$\tilde{r} = \tfrac{1}{2}Q^2/(M - \beta h); \tag{30}$$

one then has at $r = \tilde{r}$ (i.e. at $\delta = 0$)

$$\Phi_+^2 = \Phi_-^2 = a^{-2}[\tilde{r}^2 + a^2 + \epsilon(G - \epsilon)\beta h\tilde{r}]. \tag{31}$$

This quantity can be $\gtrless 1$ depending on the parameters; it is easy to show that when $\tilde{r} > 0$ (i.e. for $h < M/\beta$ when $\beta > 0$, and for any h when $\beta > 0$, see figure 3) one has

$$\Phi_\pm^2(r = \tilde{r}) \leqslant 1 \qquad \text{for } Q^2 \leqslant -2\epsilon\beta h(M - \beta h)(G - \epsilon). \tag{32}$$

This means that, unlike in the uncharged case, the solution Φ_+^2 cannot be ruled out as it can be less than one.

Note that for $\beta > 0$ and $h < M/\beta$ we have $\tilde{r} < 0$. It is moreover easy to see that the value of $\Phi_\pm^2$ at $r = 0$ does not depend on β; one of the two solutions $\Phi_\pm^2$ $(r = 0)$ is always greater than one, while for the other it is (see figure 2(*a*))

$$\Phi_-^2(r = 0) \gtreqless 0 \qquad \text{for } G^2 \gtreqless 1 + a^2/Q^2. \tag{33}$$

With the help of figure 3 and conditions (31), (32) and (33) it is now possible to draw the qualitative behaviour for the repulsive barriers also when $\beta \neq 0$. The most interesting case is when $\beta < 0$ because one can see how the gravitational repulsion and the electrostatic attraction are competing. For instance, in figure 4 the repulsive barrier

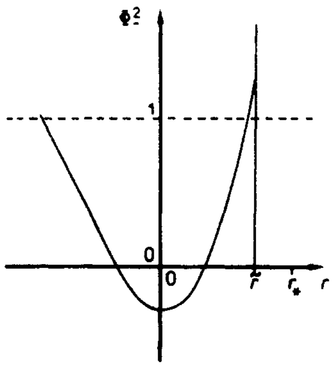

Figure 4. An example of the radial barriers for $\beta \neq 0$. See text for details.

is shown for $\beta<0$, $l<0$, $Q^2>[-2\epsilon\beta h(M-\beta h)(G-\epsilon)]$, $G^2<(1+a^2/Q^2)$. Therefore repulsive barriers also exist for particles oppositely charged relative to the source. These particles can, as expected, approach nearer to the $r=0$ disc (with respect to the uncharged case), but ultimately the gravitational repulsion prevails.

5. Conclusions

Repulsive phenomena are a peculiar feature of general relativity which has no Newtonian analogue. To get a better insight, let us evaluate the spatial velocity v of the test particle as measured by a static observer:

$$u^i=\delta^i_0/(-g_{00})^{1/2}, \qquad u^iu_i=-1 \tag{34}$$

(recall that this observer is defined only outside the ergosphere where $g_{00}<0$). We have

$$v^2=v^iv_i \qquad \text{with } v^i=-h^i_jK^j/u^iK_i \tag{35}$$

where $h_{ij}=g_{ij}+u_iu_j$ and $K^i=\mathrm{d}x^i/\mathrm{d}\lambda$ is given by (3)–(6); we limit our considerations to uncharged particles. After some algebra one has

$$v^2=\frac{1}{E^2}\left(\Gamma+\frac{2Mr-Q^2}{\Sigma}\right). \tag{36}$$

The maximum value of the velocity is attained on the surface

$$r=r_*+(r_*^2+a^2\cos^2\theta)^{1/2}. \tag{37}$$

Below this surface the velocity decreases until the particle inverts its motion; it behaves as if a new source of gravity acted upon it from above the surface (37). This has been shown to be true in the Reissner–Nordström case (Cohen and Gautreau 1979), the source being contributed by the effective mass of the electric field generated by the charge Q of the source. It is more difficult to explain the repulsive effects in the Kerr case (Israel 1970) and in the Kerr–Newman case, although we believe that a similar interpretation holds.

Acknowledgments

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